On perturbations of Dirac operators with variable magnetic field of constant direction

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Abstract

We carry out the spectral analysis of matrix valued perturbations of 3-dimensional Dirac operators with variable magnetic field of constant direction. Under suitable assumptions on the magnetic field and on the pertubations, we obtain a limiting absorption principle, we prove the absence of singular continuous spectrum in certain intervals and state properties of the point spectrum. Various situations, for example when the magnetic field is constant, periodic or diverging at infinity, are covered. The importance of an internal-type operator (a 2-dimensional Dirac operator) is also revealed in our study. The proofs rely on commutator methods.

1 Introduction and main results

We consider a relativistic spin-$\frac{1}{2}$ particle evolving in $\mathbb{R}^3$ in presence of a variable magnetic field of constant direction. By virtue of the Maxwell equations, we may assume with no loss of generality that the magnetic field has the form $\vec{B}(x_1, x_2, x_3) = \big(0, 0, B(x_1, x_2)\big)$. So the unperturbed system is described, in the Hilbert space $\mathsf{L}^2(\mathbb{R}^3; \mathbb{C}^4)$, by the Dirac operator

$$H_0 := \alpha_1 \Pi_1 + \alpha_2 \Pi_2 + \alpha_3 P_3 + \beta m,$$

where $\beta \equiv \alpha_0, \alpha_1, \alpha_2, \alpha_3$ are the usual Dirac-Pauli matrices, m is the strictly positive mass of the particle and $\Pi_j := -i\partial_j - a_j$ are the generators of the magnetic translations with a vector potential $\vec{a}(x_1, x_2, x_3) = (a_1(x_1, x_2), a_2(x_1, x_2), 0)$ that satisfies $B = \partial_1 a_2 - \partial_2 a_1$. Since $a_3 = 0$, we have written $P_3 := -i\partial_3$ instead of Π_3.

In this paper we study the stability of certain parts of the spectrum of H_0 under matrix valued perturbations V. More precisely, if V satisfies some natural hypotheses, we shall prove the absence of singular continuous spectrum and the finiteness of the point spectrum of $H := H_0 + V$ in intervals of $\mathbb{R}$ corresponding to gaps in the symmetrized spectrum of the operator $H^0 := \sigma_1 \Pi_1 + \sigma_2 \Pi_2 + \sigma_3 m$ in $\mathsf{L}^2(\mathbb{R}^2; \mathbb{C}^2)$. The matrices σ_j are the Pauli matrices and the symmetrized spectrum $\sigma^0_{\rm sym}$ of H^0 is the

union of the spectra of H^0 and $-H^0$. We stress that our analysis does not require any restriction on the behaviour of the magnetic field at infinity. Nevertheless, the pertinence of our work depends on a certain property of the internal-type operator H^0; namely, the size and the number of gaps in $\sigma^0_{\rm sym}$. We refer to [2], [7], [10], [11] and [15] for various results on the spectrum of H^0, especially in the situations of physical interest, for example when B is constant, periodic or diverges at infinity.

Technically, this work relies on commutator methods initiated by E. Mourre [13] and extensively developed in [1]. For brevity we shall constantly refer to the latter reference for notations and definitions. Our choice of a conjugate operator enables us to treat Dirac operators with general magnetic fields provided they point in a constant direction. On the other hand, as already put into evidence in [9], the use of a conjugate operator with a matrix structure has a few "rather awkward consequences" for long-range perturbations. We finally mention that this study is the counterpart for Dirac operators of [12], where only Schrödinger operators are considered. Unfortunately, the intrinsic structure of the Dirac equation prevents us from using the possible magnetic anisotropy to control the perturbations (see Remark 3.2 for details).

We give now a more precise description of our results. For simplicity we impose the continuity of the magnetic field and avoid perturbations with local singularities. Hence we assume that B is a $C(\mathbb{R}^2;\mathbb{R})$-function and choose any vector potential $\vec{a} = (a_1, a_2, 0) \in C(\mathbb{R}^2;\mathbb{R}^3)$, *e.g.* the one obtained by means of the transversal gauge [15]. The definitions below concern the admissible perturbations. In the long-range case, we restrict them to the scalar type in order not to impose unsatisfactory constraints. In the sequel, $\mathscr{B}_{\rm h}(\mathbb{C}^4)$ stands for the set of 4×4 hermitian matrices, and $\|\cdot\|$ denotes the norm of the Hilbert space $\mathcal{H} := \mathsf{L}^2(\mathbb{R}^3;\mathbb{C}^4)$ as well as the norm of $\mathscr{B}(\mathcal{H})$, the set of bounded linear operators on $\mathcal{H}$. $\mathbb{N} := \{0, 1, 2, \ldots\}$ is the set of natural numbers. ϑ is an arbitrary $C^\infty\big([0,\infty)\big)$-function such that $\vartheta = 0$ near 0 and $\vartheta = 1$ near infinity. Q_j is the multiplication operator by the coordinate x_j in $\mathcal{H}$, and the expression $\langle\cdot\rangle$ corresponds to $\sqrt{1+(\cdot)^2}$.

Definition 1.1. *Let V be a multiplication operator associated with an element of $\mathsf{L}^\infty\big(\mathbb{R}^3;\mathscr{B}_{\rm h}(\mathbb{C}^4)\big)$.*

(a) *V is* small at infinity *if* $\displaystyle\lim_{r\to\infty}\Big\|\vartheta\Big(\frac{\langle Q\rangle}{r}\Big)V\Big\| = 0,$

(b) *V is* short-range *if* $\displaystyle\int_1^\infty \Big\|\vartheta\Big(\frac{\langle Q_3\rangle}{r}\Big)V\Big\|\,\mathrm{d}r < \infty,$

(c) *Let $V_{\rm L}$ be in $C^1(\mathbb{R}^3;\mathbb{R})$ with $x \mapsto \langle x_3\rangle(\partial_j V_{\rm L})(x)$ in $\mathsf{L}^\infty(\mathbb{R}^3;\mathbb{R})$ for $j = 1,2,3$, then $V := V_{\rm L}$ is* long-range *if*

$$\int_1^\infty \Big\|\vartheta\Big(\frac{\langle Q_3\rangle}{r}\Big)\langle Q_3\rangle(\partial_j V)\Big\|\,\frac{\mathrm{d}r}{r} < \infty \qquad \textit{for} \qquad j = 1,2,3.$$

Note that Definitions 1.1.(b) and 1.1.(c) differ from the standard ones: the decay rate is imposed only in the x_3 direction.

We are in a position to state our results. Let $\mathcal{D}(\langle Q_3\rangle)$ denote the domain of $\langle Q_3\rangle$ in $\mathcal{H}$, then the limiting absorption principle for H is expressed in terms of the Banach space $\mathcal{G} := \big(\mathcal{D}(\langle Q_3\rangle), \mathcal{H}\big)_{1/2,1}$ defined by real interpolation [1]. For convenience, we recall that $\mathcal{D}(\langle Q_3\rangle^s)$ is contained in $\mathcal{G}$ for each $s > 1/2$.

Theorem 1.2. *Assume that B belongs to $C(\mathbb{R}^2;\mathbb{R})$, and that V belongs to $\mathsf{L}^\infty\big(\mathbb{R}^3;\mathscr{B}_{\mathrm{h}}(\mathbb{C}^4)\big)$, is small at infinity and can be written as the sum of a short-range and a long-range matrix valued function. Then*

(a) The point spectrum of the operator H in $\mathbb{R}\setminus\sigma^0_{\mathrm{sym}}$ is composed of eigenvalues of finite multiplicity and with no accumulation point in $\mathbb{R}\setminus\sigma^0_{\mathrm{sym}}$.

(b) The operator H has no singular continuous spectrum in $\mathbb{R}\setminus\sigma^0_{\mathrm{sym}}$.

(c) The limits $\lim_{\varepsilon\to+0}\langle\psi,(H-\lambda\mp i\varepsilon)^{-1}\psi\rangle$ exist for each $\psi\in\mathcal{G}$, uniformly in λ on each compact subset of $\mathbb{R}\setminus\{\sigma^0_{\mathrm{sym}}\cup\sigma_{\mathrm{pp}}(H)\}$.

The above statements seem to be new for such a general magnetic field. In the special but important case of a nonzero constant magnetic field B_0, the admissible perturbations introduced in Definition 1.1 are more general than those allowed in [16]. We stress that in this situation σ^0_{sym} is equal to $\{\pm\sqrt{2nB_0+m^2}\ :\ n\in\mathbb{N}\}$, which implies that there are plenty of gaps where our analysis gives results. On the other hand, if $B(x_1,x_2)\to 0$ as $|(x_1,x_2)|\to\infty$, our treatment gives no information since both $(-\infty,-m]$ and $[m,\infty)$ belong to σ^0_{sym}. We finally mention the paper [3] for a related work on perturbations of magnetic Dirac operators.

2 Mourre estimate for the operator H_0

2.1 Preliminaries

Let us start by recalling some known results. The operator H_0 is essentially self-adjoint on $\mathscr{D}:=C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$ [5, Thm. 2.1]. Its spectrum is symmetric with respect to 0 and does not contain the interval $(-m,m)$ [15, Cor. 5.14]. Thus the subset $H_0\mathscr{D}$ is dense in $\mathcal{H}$ since $\mathscr{D}$ is dense in $\mathcal{D}(H_0)$ (endowed with the graph topology) and H_0 is a homeomorphism from $\mathcal{D}(H_0)$ onto $\mathcal{H}$.

We now introduce a suitable representation of the Hilbert space $\mathcal{H}$. We consider the partial Fourier transformation

$$\mathscr{F}:\mathscr{D}\to\int_{\mathbb{R}}^{\oplus}\mathcal{H}_{12}\,\mathrm{d}\xi,\qquad(\mathscr{F}\psi)(\xi):=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathrm{e}^{-i\xi x_3}\psi(\cdot,x_3)\,\mathrm{d}x_3,\tag{2.1}$$

where $\mathcal{H}_{12}:=\mathsf{L}^2(\mathbb{R}^2;\mathbb{C}^4)$. This map extends uniquely to a unitary operator from $\mathcal{H}$ onto $\int_{\mathbb{R}}^{\oplus}\mathcal{H}_{12}\,\mathrm{d}\xi$, which we denote by the same symbol $\mathscr{F}$. As a first application, one obtains the following direct integral decomposition of H_0:

$$\mathscr{F}H_0\mathscr{F}^{-1}=\int_{\mathbb{R}}^{\oplus}H_0(\xi)\,\mathrm{d}\xi,$$

where $H_0(\xi)$ is a self-adjoint operator in $\mathcal{H}_{12}$ acting as $\alpha_1\Pi_1+\alpha_2\Pi_2+\alpha_3\xi+\beta m$ on $C_0^\infty(\mathbb{R}^2;\mathbb{C}^4)$. In the following remark we draw the connection between the operator $H_0(\xi)$ and the operator H^0 introduced in Section 1. It reveals the importance of the internal-type operator H^0 and shows why its negative $-H_0$ also has to be taken into account.

Remark 2.1. *The operator $H_0(0)$ acting on $C_0^\infty(\mathbb{R}^2;\mathbb{C}^4)$ is unitarily equivalent to the direct sum operator $\begin{pmatrix} m & \Pi_- \\ \Pi_+ & -m\end{pmatrix}\oplus\begin{pmatrix} m & \Pi_+ \\ \Pi_- & -m\end{pmatrix}$ acting on $C_0^\infty(\mathbb{R}^2;\mathbb{C}^2)\oplus C_0^\infty(\mathbb{R}^2;\mathbb{C}^2)$, where $\Pi_\pm:=\Pi_1\pm i\Pi_2$.*

Now, these two matrix operators act in $\mathsf{L}^2(\mathbb{R}^2;\mathbb{C}^2)$ and are essentially self-adjoint on $C_0^\infty(\mathbb{R}^2;\mathbb{C}^2)$ [5, Thm. 2.1]. However, the first one is nothing but H^0, while the second one is unitarily equivalent to $-H^0$ (this can be obtained by means of the abstract Foldy-Wouthuysen transformation [15, Thm. 5.13]). Therefore $H_0(0)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2;\mathbb{C}^4)$ and

$$\sigma[H_0(0)] = \sigma(H^0) \cup \sigma(-H^0) \equiv \sigma^0_{\mathrm{sym}}.$$

Moreover, there exists a relation between $\sigma[H_0(\xi)]$ and σ^0_{sym}. Indeed, for $\xi \in \mathbb{R}$ fixed, one can show that $H_0(\xi)^2 = H_0(0)^2 + \xi^2$ on $\mathcal{D}\big(H_0(\xi)^2\big) = \mathcal{D}\big(H_0(0)^2\big)$, so that

$$\sigma[H_0(\xi)^2] = \sigma[H_0(0)^2 + \xi^2] = \big(\sigma[H_0(0)]\big)^2 + \xi^2 = (\sigma^0_{\mathrm{sym}})^2 + \xi^2, \tag{2.2}$$

where the spectral theorem has been used for the second equality. Since the spectrum of $H_0(\xi)$ is symmetric with respect to 0 [15, Cor. 5.14], it follows that

$$\sigma[H_0(\xi)] = -\sqrt{(\sigma^0_{\mathrm{sym}})^2 + \xi^2} \,\cup\, \sqrt{(\sigma^0_{\mathrm{sym}})^2 + \xi^2}.$$

Define $\mu_0 := \inf|\sigma^0_{\mathrm{sym}}|$ (which is bigger or equal to m because H^0 has no spectrum in $(-m,m)$ [15, Cor. 5.14]). Then from the direct integral decomposition of H_0, one readily gets

$$\sigma(H_0) = (-\infty, -\mu_0] \cup [\mu_0, +\infty). \tag{2.3}$$

We conclude the section by giving two technical lemmas in relation with the operator H_0^{-1}. Proofs can be found in an appendix.

Lemma 2.2. *(a) For each $n \in \mathbb{N}$, $H_0^{-n}\mathscr{D}$ belongs to $\mathcal{D}(Q_3)$,*

(b) $P_3H_0^{-1}$ is a bounded self-adjoint operator equal to $H_0^{-1}P_3$ on $\mathcal{D}(P_3)$. In particular, $H_0^{-1}\mathcal{H}$ belongs to $\mathcal{D}(P_3)$.

One may observe that, given a $C^1(\mathbb{R};\mathbb{C})$-function f with f' bounded, the operator $f(Q_3)$ is well-defined on $\mathcal{D}(Q_3)$. Thus $f(Q_3)H_0^{-n}\mathscr{D}$ is a subset of $\mathcal{H}$ for each $n \in \mathbb{N}$. The preceding lemma and the following simple statement are constantly used in the sequel.

Lemma 2.3. *Let f be in $C^1(\mathbb{R};\mathbb{C})$ with f' bounded, and $n \in \mathbb{N}$. Then*

(a) $iH_0^{-1}f(Q_3) - if(Q_3)H_0^{-1}$ is equal to $-H_0^{-1}\alpha_3 f'(Q_3)H_0^{-1}$ on $H_0^{-n}\mathscr{D}$,

(b) $P_3H_0^{-1}f(Q_3) - f(Q_3)P_3H_0^{-1}$ is equal to $i(P_3H_0^{-1}\alpha_3 - 1)f'(Q_3)H_0^{-1}$ on $\mathscr{D}$.

Both right terms belong to $\mathscr{B}(\mathcal{H})$. For shortness we shall denote them by $[iH_0^{-1}, f(Q_3)]$ and $[P_3H_0^{-1}, f(Q_3)]$ respectively.

2.2 The conjugate operator

The aim of the present section is to define an appropriate operator conjugate to H_0. To begin with, one observes that $Q_3P_3H_0^{-1}\mathscr{D} \subset \mathcal{H}$ as a consequence of Lemma 2.2. In particular, the formal expression

$$A := \tfrac{1}{2}(H_0^{-1}P_3Q_3 + Q_3P_3H_0^{-1}) \tag{2.4}$$

leads to a well-defined symmetric operator on $\mathscr{D}$.

Proposition 2.4. *The operator A is essentially self-adjoint on $\mathscr{D}$ and its closure is essentially self-adjoint on any core for $\langle Q_3\rangle$.*

Proof. The claim is a consequence of Nelson's criterion of essential self-adjointness [14, Thm. X.37] applied to the triple $\{\langle Q_3\rangle, A, \mathscr{D}\}$. Let us simply verify the two hypotheses of that theorem. By using Lemmas 2.2 and 2.3, one first obtains that for all $\psi \in \mathscr{D}$:

$$\|A\psi\| = \left\|\left(P_3 H_0^{-1} Q_3 - \tfrac{1}{2}\left[P_3 H_0^{-1}, Q_3\right]\right)\psi\right\| \leq \mathrm{C}\,\|\langle Q_3\rangle\psi\|$$

for some constant $\mathrm{C} > 0$ independent of ψ. Then, for all $\psi \in \mathscr{D}$ one has:

$$\begin{aligned}\langle A\psi, \langle Q_3\rangle\psi\rangle - \langle\langle Q_3\rangle\psi, A\psi\rangle &= i\,\mathsf{Im}\left\langle Q_3\psi, \left[P_3 H_0^{-1}, \langle Q_3\rangle\right]\psi\right\rangle \\ &= i\,\mathsf{Re}\left\langle\left(\alpha_3 P_3 H_0^{-1} - 1\right)Q_3\psi, Q_3\langle Q_3\rangle^{-1}H_0^{-1}\psi\right\rangle.\end{aligned}$$

A few more commutator calculations, using again Lemma 2.3 with $f(Q_3) = \langle Q_3\rangle^{1/2}$, lead to the following result: for all $\psi \in \mathscr{D}$, there exists a constant $\mathrm{D} > 0$ independent of ψ such that

$$|\,\langle A\psi, \langle Q_3\rangle\,\psi\rangle - \langle\langle Q_3\rangle\,\psi, A\psi\rangle\,| \leq \mathrm{D}\left\|\,\langle Q_3\rangle^{\frac{1}{2}}\,\psi\right\|^2.$$ □

As far as we know, the matrix conjugate operator (2.4) has never been employed before for the study of magnetic Dirac operators.

2.3 Strict Mourre estimate for H_0

We now gather some results on the regularity of H_0 with respect to A. We recall that $\mathcal{D}(H_0)^*$ is the adjoint space of $\mathcal{D}(H_0)$ and that one has the continuous dense embeddings $\mathcal{D}(H_0) \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}(H_0)^*$, where $\mathcal{H}$ is identified with its adjoint through the Riesz isomorphism.

Proposition 2.5. *(a) The quadratic form $\mathcal{D}(A) \ni \psi \mapsto \langle H_0^{-1}\psi, iA\psi\rangle - \langle A\psi, iH_0^{-1}\psi\rangle$ extends uniquely to the bounded form defined by the operator $-H_0^{-1}(P_3 H_0^{-1})^2 H_0^{-1} \in \mathscr{B}(\mathcal{H})$.*

(b) The group $\{e^{itA}\}_{t\in\mathbb{R}}$ leaves $\mathcal{D}(H_0)$ invariant.

(c) The quadratic form

$$\mathcal{D}(A) \ni \psi \mapsto \langle H_0^{-1}(P_3 H_0^{-1})^2 H_0^{-1}\psi, iA\psi\rangle - \langle A\psi, iH_0^{-1}(P_3 H_0^{-1})^2 H_0^{-1}\psi\rangle, \tag{2.5}$$

extends uniquely to a bounded form on $\mathcal{H}$.

In the framework of [1], the statements of (a) and (c) mean that H_0 is of class $C^1(A)$ and $C^2(A)$ respectively.

Proof. (a) For any $\psi \in \mathscr{D}$, one gets

$$\begin{aligned}2\left(\langle H_0^{-1}\psi, iA\psi\rangle - \langle A\psi, iH_0^{-1}\psi\rangle\right) &= \langle[iH_0^{-1}, Q_3]\psi, P_3 H_0^{-1}\psi\rangle + \langle P_3 H_0^{-1}\psi, [iH_0^{-1}, Q_3]\psi\rangle \\ &= -\langle H_0^{-1}\psi, (\alpha_3 P_3 H_0^{-1} + H_0^{-1}\alpha_3 P_3)H_0^{-1}\psi\rangle,\end{aligned} \tag{2.6}$$

where we have used Lemmas 2.2 and 2.3 . Furthermore, one has

$$H_0^{-1}\alpha_3 = -\alpha_3 H_0^{-1} + 2H_0^{-1}P_3H_0^{-1} \tag{2.7}$$

as an operator identity in $\mathscr{B}(\mathcal{H})$. When inserting (2.7) into (2.6), one obtains the equality

$$\langle H_0^{-1}\psi, iA\psi\rangle - \langle A\psi, iH_0^{-1}\psi\rangle = -\langle \psi, H_0^{-1}(P_3H_0^{-1})^2H_0^{-1}\psi\rangle. \tag{2.8}$$

Since $\mathscr{D}$ is a core for A, the statement is obtained by density. We shall write $[iH_0^{-1}, A]$ for the bounded extension of the quadratic form $\mathcal{D}(A) \ni \psi \mapsto \langle H_0^{-1}\psi, iA\psi\rangle - \langle A\psi, iH_0^{-1}\psi\rangle$.

(b) Since $\mathcal{D}(H_0)$ is not explicitly known, one has to invoke an abstract result in order to show the invariance. Let $[iH_0, A]$ be the operator in $\mathscr{B}\left(\mathcal{D}(H_0), \mathcal{D}(H_0)^*\right)$ associated with the unique extension to $\mathcal{D}(H_0)$ of the quadratic form $\psi \mapsto \langle H_0\psi, iA\psi\rangle - \langle A\psi, iH_0\psi\rangle$ defined for all $\psi \in \mathcal{D}(H_0)\cap\mathcal{D}(A)$. Then $\mathcal{D}(H_0)$ is invariant under $\{e^{itA}\}_{t\in\mathbb{R}}$ if H_0 is of class $C^1(A)$ and if $[iH_0, A]\mathcal{D}(H_0) \subset \mathcal{H}$ [8, Lemma 2]. From equation (2.8) and [1, Eq. 6.2.24], one obtains the following equalities valid in form sense on $\mathcal{H}$:

$$-H_0^{-1}(P_3H_0^{-1})^2H_0^{-1} = \left[iH_0^{-1}, A\right] = -H_0^{-1}\left[iH_0, A\right]H_0^{-1}.$$

Thus $[iH_0, A]$ and $(P_3H_0^{-1})^2$ are equal as operators in $\mathscr{B}\left(\mathcal{D}(H_0), \mathcal{D}(H_0)^*\right)$. But since the latter belongs to $\mathscr{B}(\mathcal{H})$, $[iH, A]\mathcal{D}(H_0)$ is included in $\mathcal{H}$.

(c) The boundedness on $\mathscr{D}$ of the quadratic form (2.5) follows by inserting (2.4) into the r.h.s. term of (2.5) and by applying repeatedly Lemma 2.3 with $f(Q_3) = Q_3$. Then one concludes by using the density of $\mathscr{D}$ in $\mathcal{D}(A)$. $\square$

From now on we shall simply denote the closure in $\mathcal{H}$ of $[iH_0, A]$ by $T = (P_3H_0^{-1})^2 \in \mathscr{B}(\mathcal{H})$. One interest of this operator is that $\mathscr{F}T\mathscr{F}^{-1}$ is boundedly decomposable [6, Prop. 3.6], more precisely:

$$\mathscr{F}T\mathscr{F}^{-1} = \int_{\mathbb{R}}^{\oplus} T(\xi)\,\mathrm{d}\xi \qquad \text{with} \qquad T(\xi) = \xi^2 H_0(\xi)^{-2} \in \mathscr{B}(\mathcal{H}_{12}).$$

In the following definition, we introduce two functions giving the optimal value to a Mourre-type inequality. Remark that slight modifications have been done with regard to the usual definition [1, Sec. 7.2.1].

Definition 2.6. *Let H be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and assume that S is a symmetric operator in $\mathscr{B}\big(\mathcal{D}(H), \mathcal{D}(H)^*\big)$. Let $E^H(\lambda;\varepsilon) := E^H\big((\lambda-\varepsilon, \lambda+\varepsilon)\big)$ be the spectral projection of H for the interval $(\lambda-\varepsilon, \lambda+\varepsilon)$. Then, for all $\lambda \in \mathbb{R}$ and $\varepsilon > 0$, we set*

$$\varrho_H^S(\lambda;\varepsilon) := \sup\left\{a \in \mathbb{R} \,:\, E^H(\lambda;\varepsilon)SE^H(\lambda;\varepsilon) \geq a\,E^H(\lambda;\varepsilon)\right\},$$
$$\varrho_H^S(\lambda) := \sup_{\varepsilon>0} \varrho_H^S(\lambda;\varepsilon).$$

Let us make three observations: the inequality $\varrho_H^S(\lambda;\varepsilon') \leq \varrho_H^S(\lambda;\varepsilon)$ holds whenever $\varepsilon' \geq \varepsilon$, $\varrho_H^S(\lambda) = +\infty$ if λ does not belong to the spectrum of H, and $\varrho_H^S(\lambda) \geq 0$ for all $\lambda \in \mathbb{R}$ if $S \geq 0$. We also mention that in the case of two self-adjoint operators H and A in $\mathcal{H}$, with H of class $C^1(A)$ and $S := [iH, A]$, the function $\varrho_H^S(\cdot)$ is equal to the function $\varrho_H^A(\cdot)$ defined in [1, Eq. 7.2.4].

Taking advantage of the direct integral decomposition of H_0 and T, one obtains for all $\lambda \in \mathbb{R}$ and $\varepsilon > 0$:

$$\varrho_{H_0}^T(\lambda;\varepsilon) = \operatorname*{ess\,inf}_{\xi\in\mathbb{R}} \varrho_{H_0(\xi)}^{T(\xi)}(\lambda;\varepsilon). \tag{2.9}$$

Now we can deduce a lower bound for $\varrho_{H_0}^T(\cdot)$.

Proposition 2.7. *One has*

$$\varrho_{H_0}^T(\lambda) \geq \inf\Big\{\frac{\lambda^2-\mu^2}{\lambda^2} \ : \ \mu \in \sigma_{\rm sym}^0 \cap [0,|\lambda|]\Big\} \tag{2.10}$$

with the convention that the infimum over an empty set is $+\infty$.

Proof. We first consider the case $\lambda \geq 0$.

(i) Recall from (2.3) that $\mu_0 \equiv \inf|\sigma_{\rm sym}^0| = \inf\{\sigma(H_0)\cap[0,+\infty)\}$. Thus, for $\lambda \in [0,\mu_0)$ the l.h.s. term of (2.10) is equal to $+\infty$, since λ does not belong to the spectrum of H_0. Hence (2.10) is satisfied on $[0,\mu_0)$.

(ii) If $\lambda \in \sigma_{\rm sym}^0$, then the r.h.s. term of (2.10) is equal to 0. However, since T is positive, $\varrho_{H_0}^T(\lambda) \geq 0$. Hence the relation (2.10) is again satisfied.

(iii) Let $0 < \varepsilon < \mu_0 < \lambda$. Direct computations using the explicit form of $T(\xi)$ and the spectral theorem for the operator $H_0(\xi)$ show that for ξ fixed, one has

$$\varrho_{H_0(\xi)}^{T(\xi)}(\lambda;\varepsilon) = \inf\Big\{\frac{\xi^2}{\rho^2} \ : \ \rho \in (\lambda-\varepsilon,\lambda+\varepsilon)\cap\sigma[H_0(\xi)]\Big\} \geq \frac{\xi^2}{(\lambda+\varepsilon)^2}. \tag{2.11}$$

On the other hand one has $\varrho_{H_0(\xi)}^{T(\xi)}(\lambda;\varepsilon) = +\infty$ if $(\lambda-\varepsilon,\lambda+\varepsilon)\cap\sigma[H_0(\xi)] = \varnothing$, and a fortiori

$$\varrho_{H_0(\xi)}^{T(\xi)}(\lambda;\varepsilon) = +\infty \qquad \text{if} \qquad \big((\lambda-\varepsilon)^2,(\lambda+\varepsilon)^2\big)\cap\sigma[H_0(\xi)^2] = \varnothing.$$

Thus, by taking into account equation (2.9), (2.11), the previous observation and relation (2.2), one obtains that

$$\varrho_{H_0}^T(\lambda;\varepsilon) \geq \operatorname{ess\,inf}\Big\{\frac{\xi^2}{(\lambda+\varepsilon)^2} \ : \ \xi^2 \in \big((\lambda-\varepsilon)^2,(\lambda+\varepsilon)^2\big) - (\sigma_{\rm sym}^0)^2\Big\}. \tag{2.12}$$

Suppose now that $\lambda \notin \sigma_{\rm sym}^0$, define $\mu := \sup\{\sigma_{\rm sym}^0 \cap [0,\lambda]\}$ and choose $\varepsilon > 0$ such that $\mu < \lambda - \varepsilon$. Then the inequality (2.12) implies that

$$\varrho_{H_0}^T(\lambda;\varepsilon) \geq \frac{(\lambda-\varepsilon)^2-\mu^2}{(\lambda+\varepsilon)^2}.$$

Hence the relation (2.10) follows from the above formula when $\varepsilon \to 0$.

For $\lambda < 0$, similar arguments lead to the inequality

$$\varrho_{H_0}^T(\lambda) \geq \inf\Big\{\frac{\lambda^2-\mu^2}{\lambda^2} \ : \ \mu \in \sigma_{\rm sym}^0 \cap [\lambda,0]\Big\}.$$

The claim is then a direct consequence of the symmetry of $\sigma_{\rm sym}^0$ with respect to 0. □

The above proposition implies that we have a strict Mourre estimate, *i.e.* $\varrho_{H_0}^T(\cdot) > 0$, on $\mathbb{R}\setminus\sigma_{\rm sym}^0$. Moreover it is not difficult to prove that $\varrho_{H_0}^T(\lambda) = 0$ whenever $\lambda \in \sigma_{\rm sym}^0$. It follows that the conjugate operator A does not allow to get spectral informations on H_0 in the subset $\sigma_{\rm sym}^0$.

3 Mourre estimate for the perturbed Hamiltonian

In the sequel, we consider the self-adjoint operator $H := H_0 + V$ with a potential V that belongs to $\mathsf{L}^\infty\big(\mathbb{R}^3; \mathscr{B}_{\rm h}(\mathbb{C}^4)\big)$. The domain of H is equal to the domain $\mathcal{D}(H_0)$ of H_0. We first give a result on the difference of the resolvents $(H-z)^{-1} - (H_0 - z)^{-1}$ and, as a corollary, we obtain the localization of the essential spectrum of H.

Proposition 3.1. *Assume that V is small at infinity. Then for all $z \in \mathbb{C} \setminus \big(\sigma(H) \cup \sigma(H_0)\big)$ the difference $(H-z)^{-1} - (H_0 - z)^{-1}$ is a compact operator. It follows in particular that $\sigma_{\rm ess}(H) = \sigma_{\rm ess}(H_0)$.*

Proof. Since V is bounded and small at infinity, it is enough to check that H_0 is locally compact [15, Sec. 4.3.4]. However, the continuity of $\vec{a}$ implies that $\mathcal{D}(H_0) \subset \mathcal{H}^{1/2}_{\rm loc}$ [4, Thm. 1.3]. Hence the statement follows by usual arguments. □

Remark 3.2. *In the study of an analogous problem for Schrödinger operators [12], the authors prove a result similar to Proposition 3.1 without assuming that the perturbation is small at infinity (it only has to be small with respect to B in a suitable sense). Their proof mainly relies on the structural inequalities $H_{\rm Sch} := \Pi_1^2 + \Pi_2^2 + P_3^2 \geq \pm B$. In the Dirac case, the counterpart of these turn out to be*

$$H_0^2 \geq 2B \cdot \mathrm{diag}(0,1,0,1) \qquad \textit{and} \qquad H_0^2 \geq -2B \cdot \mathrm{diag}(1,0,1,0),$$

where $\mathrm{diag}(\cdots)$ stands for a diagonal matrix. If we assume that the magnetic field is bounded from below, the first inequality enables us to treat pertubations of the type $\mathrm{diag}(V_1, V_2, V_3, V_4)$ with V_2, V_4 small with respect to the magnetic field and V_1, V_3 small at infinity in the original sense. If the magnetic field is bounded from above, the second inequality has to be used and the role of V_2, V_4 and V_1, V_3 are interchanged. However the unnatural character of these perturbations motivated us not to include their treatment in this paper.

In order to obtain a limiting absorption principle for H, one has to invoke some abstract results. An optimal regularity condition of H with respect to A has to be satisfied. We refer to [1, Chap. 5] for the definitions of $\mathscr{C}^{1,1}(A)$ and $\mathscr{C}^{1,1}(A; \mathcal{D}(H_0), \mathcal{D}(H_0)^*)$, and for more explanations on regularity conditions.

Proposition 3.3. *Let V be a short-range or a long-range potential. Then H is of class $\mathscr{C}^{1,1}(A)$.*

Proof. Since $\{e^{itA}\}_{t\in\mathbb{R}}$ leaves $\mathcal{D}(H) = \mathcal{D}(H_0)$ invariant, it is equivalent to prove that H belongs to $\mathscr{C}^{1,1}(A; \mathcal{D}(H_0), \mathcal{D}(H_0)^*)$ [1, Thm. 6.3.4.(b)]. But in Proposition 2.5.(c), it has already been shown that H_0 is of class $C^2(A)$, so that H_0 is of class $\mathscr{C}^{1,1}(A; \mathcal{D}(H_0), \mathcal{D}(H_0)^*)$. Thus it is enough to prove that V belongs to $\mathscr{C}^{1,1}(A; \mathcal{D}(H_0), \mathcal{D}(H_0)^*)$. In the short-range case, we shall use [1, Thm. 7.5.8], which implies that V belongs to $\mathscr{C}^{1,1}(A; \mathcal{D}(H_0), \mathcal{D}(H_0)^*)$. The conditions needed for that theorem are obtained in points (i) and (ii) below. In the long-range case, the claim follows by [1, Thm. 7.5.7], which can be applied because of points (i), (iii), (iv) and (v) below.

(i) We first check that $\{e^{it\langle Q_3\rangle}\}_{t\in\mathbb{R}}$ is a polynomially bounded C_0-group in $\mathcal{D}(H_0)$ and in $\mathcal{D}(H_0)^*$. Lemma 2.3.(a) (with $n = 0$ and $f(Q_3) = \langle Q_3\rangle$) implies that H_0 is of class $C^1(\langle Q_3\rangle)$. Furthermore, by an argument similar to that given in part (b) of the proof of Proposition 2.5, one shows that $\{e^{it\langle Q_3\rangle}\}_{t\in\mathbb{R}}$ leaves $\mathcal{D}(H_0)$ invariant. Since $H_0 e^{it\langle Q_3\rangle} - e^{it\langle Q_3\rangle} H_0$, defined on $\mathscr{D}$, extends continuously to the operator $t\alpha_3 Q_3 \langle Q_3\rangle^{-1} e^{it\langle Q_3\rangle} \in \mathscr{B}(\mathcal{H})$, one gets that $\|e^{it\langle Q_3\rangle}\|_{\mathscr{B}(\mathcal{D}(H_0))} \leq$ Const.$\langle t\rangle$ for all $t \in \mathbb{R}$,

*i.e.*the polynomial bound of the C_0-group in $\mathcal{D}(H_0)$. By duality, $\{e^{it\langle Q_3\rangle}\}_{t\in\mathbb{R}}$ extends to a polynomially bounded C_0-group in $\mathcal{D}(H_0)^*$ [1, Prop. 6.3.1]. The generators of these C_0-groups are densely defined and closed in $\mathcal{D}(H_0)$ and in $\mathcal{D}(H_0)^*$ respectively; both are simply denoted by $\langle Q_3\rangle$.

(ii) Since $\{e^{itA}\}_{t\in\mathbb{R}}$ leaves $\mathcal{D}(H_0)$ invariant, one may also consider the C_0-group in $\mathcal{D}(H_0)$ obtained by restriction and the C_0-group in $\mathcal{D}(H_0)^*$ obtained by extension. The generator of each of these C_0-groups will be denoted by A. Let $\mathcal{D}\big(A;\mathcal{D}(H_0)\big) := \{\varphi\in\mathcal{D}(H_0)\cap\mathcal{D}(A)\ :\ A\varphi\in\mathcal{D}(H_0)\}$ be the domain of A in $\mathcal{D}(H_0)$, and let $\mathcal{D}\big(A^2;\mathcal{D}(H_0)\big) := \{\varphi\in\mathcal{D}(H_0)\cap\mathcal{D}(A^2)\ :\ A\varphi, A^2\varphi\in\mathcal{D}(H_0)\}$ be the domain of A^2 in $\mathcal{D}(H_0)$. We now check that $\langle Q_3\rangle^{-1}A$ and $\langle Q_3\rangle^{-2}A^2$, defined on $\mathcal{D}\big(A;\mathcal{D}(H_0)\big)$ and on $\mathcal{D}\big(A^2;\mathcal{D}(H_0)\big)$ respectively, extend to operators in $\mathscr{B}\big(\mathcal{D}(H_0)\big)$. After some commutator calculations performed on $\mathscr{D}$ and involving Lemma 2.3, one first obtains that $\langle Q_3\rangle^{-1}A$ and $\langle Q_3\rangle^{-2}A$ are respectively equal on $\mathscr{D}$ to some operators S_1 and $S_2\langle Q_3\rangle^{-1}$ in $\mathscr{B}(\mathcal{H})$, where S_1 and S_2 are polynomials in H_0^{-1}, $P_3H_0^{-1}$, α_3 and $f(Q_3)$ for bounded functions f with bounded derivatives. Since $\mathscr{D}$ is a core for A, these equalities even hold on $\mathcal{D}(A)$. Hence one has on $\mathcal{D}(A^2)$:

$$\langle Q_3\rangle^{-2}A^2 = \big(\langle Q_3\rangle^{-2}A\big)\,A = S_2\langle Q_3\rangle^{-1}A = S_2S_1\,.$$

In consequence, $\langle Q_3\rangle^{-1}A$ and $\langle Q_3\rangle^{-2}A^2$ are equal on $\mathcal{D}(A)$ and on $\mathcal{D}(A^2)$ respectively, to operators expressed only in terms of H_0^{-1}, $P_3H_0^{-1}$, α_3 and $f(Q_3)$ for bounded functions f with bounded derivatives. Moreover, one easily observes that these operators and their products belong to $\mathscr{B}\big(\mathcal{D}(H_0)\big)$. Thus, it follows that $\langle Q_3\rangle^{-1}A$ and $\langle Q_3\rangle^{-2}A^2$ are equal on $\mathcal{D}\big(A;\mathcal{D}(H_0)\big)$ and on $\mathcal{D}\big(A^2;\mathcal{D}(H_0)\big)$ respectively to some operators belonging to $\mathscr{B}\big(\mathcal{D}(H_0)\big)$.

(iii) By duality, the operator $(\langle Q_3\rangle^{-1}A)^*$ belongs to $\mathscr{B}\big(\mathcal{D}(H_0)^*\big)$. Now, for $\psi\in\mathcal{D}(H_0)^*$ and $\varphi\in\mathcal{D}\big(A;\mathcal{D}(H_0)\big)$, one has

$$\langle(\langle Q_3\rangle^{-1}A)^*\psi,\varphi\rangle = \langle\psi,\langle Q_3\rangle^{-1}A\varphi\rangle = \langle\langle Q_3\rangle^{-1}\psi,A\varphi\rangle, \tag{3.13}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between $\mathcal{D}(H_0)$ and $\mathcal{D}(H_0)^*$. Since $\langle Q_3\rangle^{-1}$ is a homeomorphism from $\mathcal{D}(H_0)^*$ to the domain of $\langle Q_3\rangle$ in $\mathcal{D}(H_0)^*$, it follows from (3.13) that the domain of $\langle Q_3\rangle$ in $\mathcal{D}(H_0)^*$ is included in the domain of A in $\mathcal{D}(H_0)^*$ (the adjoint of the operator A in $\mathcal{D}(H_0)$ is equal to the operator $-A$ in $\mathcal{D}(H_0)^*$).

(iv) The inequality $r\|(\langle Q_3\rangle+ir)^{-1}\|_{\mathscr{B}(\mathcal{D}(H_0)^*)}\le$ Const. for all $r>0$ is obtained from relation (3.15), given in the proof of Lemma 2.3, with $f(Q_3)=(\langle Q_3\rangle+ir)^{-1}$.

(v) Assume that V is a long-range (scalar) potential. Then the following equality holds in form sense on $\mathscr{D}$:

$$2\,[iV,A] = -Q_3(\partial_3V)H_0^{-1} - H_0^{-1}Q_3(\partial_3V) + \big[iV,H_0^{-1}\big]\,Q_3P_3 + P_3Q_3\,\big[iV,H_0^{-1}\big]\,, \tag{3.14}$$

with $\big[iV,H_0^{-1}\big]=\sum_{j=1}^3 H_0^{-1}\alpha_j(\partial_jV)H_0^{-1}$. Using Lemma 2.3.(a), one gets that the last two terms in (3.14) are equal in form sense on $\mathscr{D}$ to

$$2\,\mathsf{Re}\sum_{j=1}^3 H_0^{-1}\alpha_jQ_3(\partial_jV)P_3H_0^{-1} - 2\,\mathsf{Im}\sum_{j=1}^3 H_0^{-1}\alpha_j(\partial_jV)H_0^{-1}\alpha_3P_3H_0^{-1}.$$

It follows that $[iV,A]$, defined in form sense on $\mathscr{D}$, extends continuously to an operator in $\mathscr{B}(\mathcal{H})$. Now let ϑ be as in Definition 1.1. Then a direct calculation using the explicit form of $[iV,A]$ obtained above

implies that

$$\Big\|\vartheta\Big(\frac{\langle Q_3\rangle}{r}\Big)\,[iV,A]\Big\| \le \mathrm{C}\sum_{j=1}^{3}\Big\|\vartheta\Big(\frac{\langle Q_3\rangle}{r}\Big)\langle Q_3\rangle(\partial_j V)\Big\| + \frac{\mathrm{D}}{r}$$

for all $r>0$ and some positive constants C and D. □

As a direct a consequence, one obtains that

Lemma 3.4. *If V satisfies the hypotheses of Theorem 1.2, then A is conjugate to H on $\mathbb{R}\setminus\sigma^0_{\rm sym}$.*

Proof. Proposition 3.3 implies that both H_0 and H are of class $\mathscr{C}^{1,1}(A)$. Furthermore, the difference $(H+i)^{-1}-(H_0+i)^{-1}$ is compact by Proposition 3.1, and $\varrho^T_{H_0}>0$ on $\mathbb{R}\setminus\sigma^0_{\rm sym}$ due to Proposition 2.7. Hence the claim follows by [1, Thm. 7.2.9 & Prop. 7.2.6]. □

We can finally give the proof of Theorem 1.2.

Proof of Theorem 1.2. Since A is conjugate to H on $\mathbb{R}\setminus\sigma^0_{\rm sym}$ by Lemma 3.4, the assertions (a) and (b) follow by the abstract conjugate operator method [1, Cor. 7.2.11 & Thm. 7.4.2].

The limiting absorption principle directly obtained via [1, Thm. 7.4.1] is expressed in terms of some interpolation space, associated with $\mathcal{D}(A)$, and of its adjoint. Since both are not standard spaces, one may use [1, Prop. 7.4.4] for the Friedrichs couple $(\mathcal{D}(\langle Q_3\rangle),\mathcal{H})$ to get the statement (c). In order to verify the hypotheses of that proposition, one has to check that for each $z\in\mathbb{C}\setminus\sigma(H)$ the inclusion $(H-z)^{-1}\mathcal{D}(\langle Q_3\rangle)\subset\mathcal{D}(A)$ holds. However, since $\mathcal{D}(\langle Q_3\rangle)$ is included in $\mathcal{D}(A)$ by Proposition 2.4, it is sufficient to prove that for each $z\in\mathbb{C}\setminus\sigma(H)$ the operator $(H-z)^{-1}$ leaves $\mathcal{D}(\langle Q_3\rangle)$ invariant. Since $\mathcal{D}(H)=\mathcal{D}(H_0)$ is left dinvariant by the group $\{e^{it\langle Q_3\rangle}\}_{t\in\mathbb{R}}$ (see Proposition 3.3 (i)) one easily gets from [1, Thm. 6.3.4.(a)] that H is of class $C^1(\langle Q_3\rangle)$, which implies the required invariance of $\mathcal{D}(\langle Q_3\rangle)$ [1, Thm. 6.2.10.(b)]. □

Appendix

Proof of Lemma 2.2. (a) Let φ, ψ be in $\mathscr{D}$. Using the transformation (2.1), one gets

$$\big\langle H_0^{-n}\varphi, Q_3\psi\big\rangle = \int_{\mathbb{R}}\big\langle H_0(\xi)^{-n}(\mathscr{F}\varphi)(\xi),(i\partial_\xi\mathscr{F}\psi)(\xi)\big\rangle_{\mathcal{H}_{12}}\,\mathrm{d}\xi.$$

Now the map $\mathbb{R}\ni\xi\mapsto H_0(\xi)^{-n}\in\mathscr{B}(\mathcal{H}_{12})$ is norm differentiable with its derivative equal to $-\sum_{j=1}^n H_0(\xi)^{-j}\alpha_3 H_0(\xi)^{j-n-1}$. Hence $\{\partial_\xi[H_0(\xi)^{-n}(\mathscr{F}\varphi)(\xi)]\}_{\xi\in\mathbb{R}}$ belongs to $\int_{\mathbb{R}}^{\oplus}\mathcal{H}_{12}\,\mathrm{d}\xi$. Thus one can perform an integration by parts (with vanishing boundary contributions) and obtain

$$\big\langle H_0^{-n}\varphi, Q_3\psi\big\rangle = \int_{\mathbb{R}}\big\langle i\partial_\xi[H_0(\xi)^{-n}(\mathscr{F}\varphi)(\xi)],(\mathscr{F}\psi)(\xi)\big\rangle_{\mathcal{H}_{12}}\,\mathrm{d}\xi.$$

It follows that $\big|\big\langle H_0^{-n}\varphi,Q_3\psi\big\rangle\big|\le \mathrm{Const.}\|\psi\|$ for all $\psi\in\mathscr{D}$. Since Q_3 is essentially self-adjoint on $\mathscr{D}$, this implies that $H_0^{-n}\varphi$ belongs to $\mathcal{D}(Q_3)$.

(b) The boundedness of $P_3 H_0^{-1}$ is a consequence of the estimate

$$\operatorname*{ess\,sup}_{\xi\in\mathbb{R}} \|\xi H_0(\xi)^{-1}\|_{\mathscr{B}(\mathcal{H}_{12})} = \operatorname*{ess\,sup}_{\xi\in\mathbb{R}} \left\| \frac{|\xi|}{[H_0(0)^2+\xi^2]^{1/2}} \right\|_{\mathscr{B}(\mathcal{H}_{12})} < \infty$$

and of the direct integral formalism [6, Prop. 3.6 & 3.7]. The remaining assertions follow by standard arguments. □

Proof of Lemma 2.3. (a) One first observes that the following equality holds on $\mathscr{D}$:

$$iH_0^{-1} f(Q_3) H_0 = -H_0^{-1}\alpha_3 f'(Q_3) + if(Q_3). \tag{3.15}$$

Now, for $\varphi, \psi \in \mathscr{D}$ and $\eta \in H_0^{-n}\mathscr{D}$, one has

$$\begin{aligned}
&\langle \varphi, iH_0^{-1} f(Q_3)\eta\rangle - \langle \varphi, if(Q_3)H_0^{-1}\eta\rangle \\
&= \langle \varphi, iH_0^{-1} f(Q_3)H_0\psi\rangle + \langle \varphi, iH_0^{-1} f(Q_3)(\eta - H_0\psi)\rangle - \langle \bar{f}(Q_3)\varphi, iH_0^{-1}\eta\rangle \\
&= -\langle \varphi, H_0^{-1}\alpha_3 f'(Q_3)H_0^{-1}\eta\rangle - \langle \varphi, H_0^{-1}\alpha_3 f'(Q_3)H_0^{-1}(H_0\psi - \eta)\rangle \\
&\quad + \langle \bar{f}(Q_3)\varphi, iH_0^{-1}(H_0\psi - \eta)\rangle + \langle \bar{f}(Q_3)H_0^{-1}\varphi, i(\eta - H_0\psi)\rangle,
\end{aligned}$$

where we have used (3.15) in the last equality for the term $\langle \varphi, iH_0^{-1} f(Q_3)H_0\psi\rangle$. Hence there exists a constant C (depending on φ) such that

$$|\langle \varphi, iH_0^{-1} f(Q_3)\eta\rangle - \langle \varphi, if(Q_3)H_0^{-1}\eta\rangle + \langle \varphi, H_0^{-1}\alpha_3 f'(Q_3)H_0^{-1}\eta\rangle| \leq \mathrm{C}\,\|\eta - H_0\psi\|.$$

Then the statement is a direct consequence of the density of $H_0\mathscr{D}$ and $\mathscr{D}$ in $\mathcal{H}$.

(b) This is a simple corollary of the point (a). □

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